

ON MAXIMAL IMMEDIATE EXTENSIONS OF VALUED FIELDS

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ABSTRACT. We study the maximal immediate extensions of valued fields whose residue fields are perfect and whose value groups are divisible by the residue characteristic if it is positive. In the case where there is such an extension which has finite transcendence degree we derive strong properties of the field and the extension and show that the maximal immediate extension is unique up to isomorphism, although these fields need not be Kaplansky fields. If the maximal immediate extension is an algebraic extension, we show that it is equal to the perfect hull and the completion of the field.

1. INTRODUCTION

In this paper, we denote a valued field by (K, v) , its value group by vK , and its residue field by Kv . When we talk of a valued field extension $(L|K, v)$ we mean that (L, v) is a valued field, $L|K$ a field extension, and K is endowed with the restriction of v . For the basic facts about valued fields, we refer the reader to [5, 6, 15, 18, 21, 22].

A **henselian field** is a valued field which satisfies Hensel's Lemma, or equivalently, admits a unique extension of its valuation to its algebraic closure. Note that every algebraic extension of a henselian field is again henselian, with respect to the unique extension of the valuation. A **henselization** of a valued field (K, v) is an algebraic extension which is henselian and minimal in the sense that it can be embedded in every other henselian extension field of (K, v) . Henselizations exist for every valued field (K, v) , and they are unique up to valuation preserving isomorphism over K . Therefore, we will speak of "the henselization of (K, v) " and denote it by (K^h, v) .

An extension $(L|K, v)$ is called **immediate** if the canonical embeddings of vK in vL and of Kv in Lv are onto, in other words, value group and residue field remain unchanged. Henselizations are immediate separable-algebraic extensions. A valued field is called **maximal** if it does not admit any nontrivial immediate extensions. It follows that a maximal immediate extension of a valued field is a maximal field. It was shown by W. Krull in [9] that every valued field (K, v) admits a maximal immediate extension (M, v) (the proof was later simplified by K. A. H. Gravett in [7]). However, the maximal immediate extension M does not need to be unique up to isomorphism. This was shown by I. Kaplansky in [8]. He proved also that under a certain condition, called "hypothesis A", uniqueness holds (see below). A valued field (K, v) satisfying hypothesis A is called a **Kaplansky field**. By Theorem 1 of [23], hypothesis A is equivalent to the conjunction of the following three conditions, where p denotes the characteristic $\text{char } Kv$ of the residue field:

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- (K1) if $p > 0$ then the value group vK is p -divisible,
- (K2) the residue field Kv is perfect,
- (K3) the residue field Kv admits no finite separable extension of degree divisible by p .

A more elementary proof for the equivalence was later given by Kaplansky himself, based on an idea of D. Leep, and is documented in [10]. Note that conditions (K2) and (K3) can be combined into the condition that the residue field Kv admits no finite extensions of degree divisible by its characteristic. But splitting this up has a purpose; see the paper [16] which presents an alternative proof of the following theorem:

Theorem 1.1 (Kaplansky, Theorem 5 of [8]). *If (K, v) is a Kaplansky field, then the maximal immediate extension of (K, v) is unique up to valuation preserving isomorphism over K .*

Note further that every valued field of residue characteristic 0 is a Kaplansky field.

We will study properties of valued fields satisfying the following conditions:

- (1) (K, v) is a henselian field such that (K1) and (K2) hold.

This includes all Kaplansky fields and in particular, all valued fields of residue characteristic 0. We will **not** assume that (K, v) satisfies (K3).

Theorem 1.2. *In addition to the assumptions (1), suppose that (M, v) is maximal, of finite transcendence degree over K and v is nontrivial on M . Take $L|K$ to be the maximal separable-algebraic subextension of $M|K$. Then we have:*

- a) K is a separably tame field,
- b) $L|K$ is a finite tame extension,
- c) the perfect hull of K is contained in the completion of K ,
- d) vM/vK and $Mv|Kv$ are finite.

If in addition, $M|K$ is algebraic, then we also have:

- e) M is equal to the perfect hull of L and to the completion of L ,
- f) the perfect hull of K is equal to the completion of K and is the unique maximal immediate extension of K .

Note that a field with the trivial valuation is maximal. Together with the above theorem this yields the following:

Corollary 1.3. *In addition to the assumptions (1), suppose that (M, v) is a maximal immediate extension of (K, v) such that $M|K$ is algebraic.*

- a) If $\text{char } K = 0$, then $M = K$, so (K, v) is maximal.
- b) Otherwise, M is equal to the perfect hull of K and to the completion of K .

Note that if in the situation of the above corollary the extension $M|K$ is separable-algebraic, then regardless of the characteristic of K we obtain that (K, v) is a maximal field. The next corollary shows that this holds also if we do not require $M|K$ to be immediate.

Corollary 1.4. *Take a valued field (K, v) satisfying the assumptions (1) with v nontrivial on K , and a separable-algebraic extension M of K . If (M, v) is maximal, then (K, v) is maximal and a tame field, and $(M|K, v)$ is a finite tame extension.*

The conclusion of this corollary remains true if we replace “separable-algebraic extension” by “finite extension”. A valued field (K, v) is called **maximal-by-finite** if it is not maximal, but a finite extension (M, v) of (K, v) is a maximal field.

Theorem 1.5. *Take a valued field (K, v) satisfying the assumptions (1), and a finite extension M of K . If (M, v) is maximal, then (K, v) is maximal and a tame field, and $(M|K, v)$ is a tame extension. Hence, a valued field which satisfies the assumptions (1) cannot be maximal-by-finite.*

This theorem constitutes a partial answer, for the case that (K, v) satisfies **(K1)** and **(K2)**, to a question we have been asked by K. Struyve and K. Kedlaya.

Maximal-by-finite fields are of importance in connection with decomposition problems for modules over valuation domains. For this purpose, the characterization of non-henselian maximal-by-finite fields has already been given by P. Vamos in [19]. His result can be formulated as follows:

Theorem 1.6. *Suppose that (K, v) is maximal-by-finite, but not henselian. Then K is formally real, Kv is algebraically closed (and hence v is not compatible with any ordering on K), v admits two distinct immediate extensions to $K(\sqrt{-1})$ and with both of them, $K(\sqrt{-1})$ is maximal. Exactly one of the following two cases holds:*

- i) K is real closed, $K(\sqrt{-1})$ is algebraically closed and the completion of (K, v) .
- ii) v is a composition $v = w \circ \bar{w}$ with both w and \bar{w} nontrivial, (K, w) is maximal and case i) holds for (Kw, \bar{w}) ; in this case, (K, v) is complete.

These two theorems leave open the case of henselian fields that violate **(K1)** or **(K2)**. Not much seems to be known in this case. M. Nagata ([17, Appendix, Example (E3.1), pp. 206-207]) gave an example of a discrete valued field which is not maximal but has a finite purely inseparable maximal immediate extension. This field has infinite p -degree and it can be shown that this fact is necessary. But there appear to be no results in the literature for non-discrete henselian fields that violate **(K1)** or **(K2)**. However, there are related results that show the influence of the p -degree on the behaviour of maximal immediate extensions: see Theorems 1.2, 1.5 and 1.6 of [4].

In the situation of Corollary 1.3, the maximal immediate extension of (K, v) is always unique up to valuation preserving isomorphism over K . This result can be generalized to the case of finite transcendence degree:

Theorem 1.7. *Assume that the assumptions (1) are satisfied. If the field admits a maximal immediate extension of finite transcendence degree, then all maximal immediate extensions of (K, v) are isomorphic over K , as valued fields. If (K, v) admits an immediate extension of infinite transcendence degree, then all maximal immediate extensions of (K, v) are of infinite transcendence degree over K .*

Theorem 1.7 shows that there are valued fields that are not maximal and not Kaplansky fields but admit unique maximal immediate extensions. To produce examples of such fields, take any valued field (K_0, v) that is not maximal and satisfies **(K1)** and **(K2)**, but not **(K3)**. Take any maximal immediate extension (M, v) of (K_0, v) and denote the transcendence basis of $M|K_0$ by T . Choose any finite subset T_0 of T . Then the henselization of $(K_0(T \setminus T_0), v)$ does not satisfy **(K3)**, but by Theorem 1.7, its maximal immediate extension is unique up to valuation preserving isomorphism.

The question under which additional assumptions the converse of Theorem 1.1 holds was studied in [20] and in [16]. Proposition 8.5 in the latter paper provides such an assumption, but it is not satisfied by the valued fields we have constructed since by part a) of Theorem 1.2 they are separably tame.

If (L, v) is a finite extension of (K, v) such that the extension of v from K to L is unique, then the **Lemma of Ostrowski** (see [22], Chapter VI, §12, Corollary to Theorem 25) says that

$$(2) \quad [L : K] = p^\nu(vL : vK)[Lv : Kv]$$

for a nonnegative integer ν and p the **characteristic exponent** of Kv , that is, $p = \text{char}Kv$ if it is positive and $p = 1$ otherwise. The factor $d(L|K, v) := p^\nu$ is called the **defect** of the extension $(L|K, v)$. If it is nontrivial, that is, if $\nu > 0$, then $(L|K, v)$ is called a **defect extension**. Otherwise, $(L|K, v)$ is called a **defectless extension**.

The following theorem relates the problem of the existence of nontrivial separable-algebraic defect extensions of (K, v) with the structure of the maximal immediate extensions of the field.

Theorem 1.8. *In addition to the assumptions (1), suppose that at least one of the following cases holds:*

- a) (K, v) admits a finite separable-algebraic defect extension,
- b) the perfect hull of K is not contained in the completion of K .

Then every maximal immediate extension of (K, v) is of infinite transcendence degree over K .

Moreover, condition b) implies condition a).

Open question: What can be said about a valued field that admits a maximal immediate extension of finite transcendence degree but violates **(K1)** or **(K2)**?

This paper is in part based on the thesis [2] of the first author.

2. PRELIMINARIES

By $\text{char } K$ we denote the characteristic of K . In slight abuse of notation, we will denote the perfect hull of a field K by K^{1/p^∞} even if in the respective context, p denotes a prime other than $\text{char } K$. If $\text{char } K = 0$, then $K^{1/p^\infty} = K$ even if K is valued with $\text{char } Kv = p > 0$.

The algebraic closure of a field K will be denoted by \tilde{K} , and its separable-algebraic closure by K^{sep} .

2.1. Tame and defect extensions. Take a finite extension $(L|K, v)$ of valued fields. If $v_1 = v, \dots, v_g$ are the distinct extensions of the valuation v of K to the field L , then $L|K$ satisfies the **fundamental inequality** (cf. Corollary 17.5 of [5]):

$$(3) \quad [L : K] \geq \sum_{i=1}^g (v_i L : vK)[Lv_i : Kv].$$

A useful consequence of the above inequality is the following fact.

Lemma 2.1. *If L is a finite extension of a valued field (K, v) , then the extension of v from K to L is unique if and only if $L|K$ is linearly disjoint from some (equivalently, every) henselization of (K, v) .*

Proof. By Corollary 7.48 of [15], we have that

$$(4) \quad [L : K] = \sum_{i=1}^g [L^{h(v_i)} : K^{h(v_i)}] = [K^{h(v_i)} \cdot L : K^{h(v_i)}],$$

where v_1, \dots, v_g are the distinct extensions of v to L and $L^{h(v_i)}, K^{h(v_i)}$ are henselizations of L and K with respect to an extension of v_i to $\tilde{L} = \tilde{K}$. If $L|K$ is not linearly disjoint from $K^{h(v_i)}|K$ for some $i \leq g$, then $[K^{h(v_i)} \cdot L : K^{h(v_i)}] < [L : K]$. Thus $g \geq 2$ and the extension of v from K to L is not unique.

On the other hand, if $L|K$ is linearly disjoint from some henselization K^h of K , then $[L : K] = [K^h \cdot L : K^h]$ and from equation (4) we deduce that v admits a unique extension from K to L . \square

Assume now that the extension of the valuation v from K to L is unique. Then inequality (3) is of the form

$$(5) \quad [L : K] \geq (vL : vK)[Lv : Kv],$$

and the “missing factor” on the right hand side of the inequality is determined by the Lemma of Ostrowski, i.e., equation (2). Fix an extension of v from K to \tilde{K} and denote it again by v . Then from Lemma 2.1 it follows that $[L : K] = [L \cdot K^h : K^h] = [L^h : K^h]$. Since $(K^h|K, v)$ and $(L^h|L, v)$ are immediate extensions, $(vL : vK)[Lv : Kv] = (vL^h : vK^h)[L^h v : K^h v]$. Together with the definition of the defect, this yields that

$$(6) \quad d(L|K, v) = d(L^h|K^h, v).$$

It follows that if $(L|K, v)$ is a defect extension, then also $(L^h|K^h, v)$ has nontrivial defect. Therefore we will restrict our studies of the defect to the case of henselian fields.

An algebraic extension $(L|K, v)$ of a henselian field (K, v) is called **tame** if every finite subextension $F|K$ of $L|K$ satisfies the following conditions:

- (T1) if $\text{char } Kv = p > 0$, then the ramification index $(vF : vK)$ is prime to p ,
- (T2) the residue field extension $Fv|Kv$ is separable,
- (T3) $(F|K, v)$ is a defectless extension.

Assume that (K, v) is a henselian field with $\text{char } Kv = 0$. Then the first two conditions of the above definition are trivially satisfied and the third one follows immediately from the Lemma of Ostrowski. Hence every algebraic extension of such a field is tame.

A henselian field (K, v) is said to be a **tame field** if $(\tilde{K}|K, v)$ is a tame extension, and a **separably tame field** if $(K^{\text{sep}}|K, v)$ is a tame extension. From the definition of a tame extension it follows that (K, v) is tame if and only if

- (TF1) if $\text{char } Kv = p > 0$, then the value group vK is p -divisible,
- (TF2) the residue field Kv is perfect,
- (TF3) (K, v) is a defectless field.

Note that by (TF1) and (TF2), the perfect hull of a tame field is an immediate extension, and by (TF3), this extension must be trivial. This shows that every tame field is perfect.

Take a valued field (K, v) , fix an extension of v to K^{sep} and call it again v . The fixed field of the closed subgroup

$$G^r := \{\sigma \in \text{Gal}(K^{\text{sep}}|K) \mid v(\sigma a - a) > va \text{ for all } a \in \mathcal{O}_{K^{\text{sep}}} \setminus \{0\}\}$$

of $\text{Gal}(K^{\text{sep}}|K)$ (cf. Corollary 20.6 of [5]) is called the **absolute ramification field of** (K, v) and is denoted by $(K, v)^r$ or K^r if v is fixed. Ramification theory states that $K^r = \tilde{K}$ if $\text{char } Kv = 0$, and $K^{\text{sep}}|K^r$ is a **p -extension** if $\text{char } Kv = p > 0$, i.e., a Galois extension with a pro- p -group as its Galois group (cf. Lemma 2.7 of [11]). Moreover, for any algebraic extension L of K we have that

$$(7) \quad L^r = K^r \cdot L$$

(cf. Theorem 4.10 of [13]). From Section 4 of [13], equation (4.5), it follows that $K^h \subseteq K^r$. In particular, together with equation (7) it shows that

$$(8) \quad (K^h)^r = K^r.$$

If (K, v) is henselian, then K^r is a Galois extension of K , and it is also the unique maximal tame extension of (K, v) (see Theorem 20.10 of [5] and Proposition 4.1 of [16]). This yields that every tame extension of valued fields is separable-algebraic. Furthermore, we obtain that (K, v) is a tame field if and only if $K^r = \tilde{K}$.

Lemma 2.2. *Take a normal extension N of a henselian field (K, v) and set $L = N \cap K^r$. Then vN/vL is a p -group, $Nv|Lv$ is purely inseparable and $(L|K, v)$ is a tame extension.*

Proof. Since $L|K$ is a subextension of the tame extension $K^r|K$, it is also tame. The assertions on value group and residue field follow from general ramification theory. \square

We now turn to a result that will be crucial for the proof of our main theorems. In order to prove it, we need the following lemma.

Lemma 2.3. *The residue field of a henselian valuation on an ordered field has characteristic 0.*

Proof. If the residue field of the henselian field (K, v) has characteristic $p > 0$, then the reduction of the polynomial $X^2 - X + p$ under v is the polynomial v is the polynomial $X^2 - X$ which has the two distinct roots 0 and 1. Hence by Hensel's Lemma, $X^2 - X + p$ splits in K , which is impossible in any ordered field. \square

Theorem 2.4. *Assume that $(L|K, v)$ is a finite extension of henselian fields. If (L, v) is a tame field, then also (K, v) is a tame field and the extension $(L|K, v)$ is defectless.*

Proof. Since $L^r = L \cdot K^r$, we know that $L^r|K^r$ is a finite extension. Since (L, v) is assumed to be a tame field, we have that L^r is equal to the algebraic closure $\tilde{L} = \tilde{K}$. By Artin-Schreier Theory, K^r is either algebraically closed or real closed. The latter is not possible. Indeed, since v is henselian on K^r , it would have residue characteristic 0 by the previous lemma. But then, (K^r, v) would be a tame field and thus algebraically closed. We conclude that K^r is algebraically closed, showing that (K, v) is a tame field. By definition, it follows that the finite extension $(L|K, v)$ is tame and hence defectless. \square

For the conclusion of this section we list two more useful results. The following is Lemma 4.15 of [14]:

Lemma 2.5. *Assume that (L, v) is a tame field and K is a relatively algebraically closed subfield of L . If in addition $Lv|Kv$ is an algebraic extension, then (K, v) is also a tame field.*

One of the ingredients in the proof of the previous lemma is the following fact, which is proved by use of Hensel's Lemma (see, e.g., Lemma 2.4 of [4]).

Lemma 2.6. *Assume (L, v) to be henselian and K to be relatively separable-algebraically closed in L . Then Kv is relatively separable-algebraically closed in Lv . If in addition $Lv|Kv$ is algebraic, then the torsion subgroup of vL/vK is a p -group, where p is the characteristic exponent of Kv .*

2.2. Defect extensions of prime degree. In the study of defect extensions, reduction to the case of defect extensions of prime degree is a crucial tool. This reduction is achieved thanks to the following important property of the absolute ramification field, which is deduced via Galois correspondence from the fact that G^r is a pro- p -group. For the proof, see Lemma 2.9 of [11].

Lemma 2.7. *Let (K, v) be a valued field extension and take p to be the characteristic exponent of Kv . Then every finite extension of K^r is a tower of normal extensions of degree p . If $L|K$ is a finite extension, then there is already a finite tame extension N of K^h such that $L.N|N$ is such a tower.*

The defect is preserved under liftings through tame extensions (see Proposition 2.8 of [11]):

Proposition 2.8. *Take a henselian field (K, v) and a tame extension N of K . Then for any finite extension $L|K$,*

$$d(L|K, v) = d(L.N|N, v).$$

Take a valued field (K, v) of positive residue characteristic p . Fix an extension of v to K^{sep} . Denote by K^h and K^r the henselization and the absolute ramification field of K with respect to this extension. Take any finite extension $(L|K, v)$ such that the extension of the valuation v from K to L is unique. Then equation (6) together with Proposition 2.8 and equation (8) give that

$$d(L|K, v) = d(L.K^h|K^h, v) = d(L.K^r|K^r, v).$$

On the other hand, Lemma 2.7 shows that $L.K^r|K^r$ is a tower of normal extensions of degree p . Thus, if $L|K$ is separable, then $L.K^r|K^r$ is a tower of Galois extensions of degree p and if $(L|K, v)$ is a defect extension, then so are some of these extensions. This shows that Galois defect extensions of prime degree play a crucial role in the investigation of defect extensions.

If $\text{char } K = p$, then every Galois extension of degree p is an **Artin-Schreier extension**, i.e., an extension generated by a root ϑ of a polynomial $X^p - X - a$ with $a \in K$. In this case, ϑ is called an **Artin-Schreier generator** of the extension. On the other hand, if a polynomial $f = X^p - X - a \in K[X]$ has no roots in K , then it is irreducible over the field. If ϑ is a root of f , then the other roots are of the form $\vartheta + 1, \dots, \vartheta + p - 1$. Hence $K(\vartheta)|K$ is a Galois extension.

Note that an Artin-Schreier extension $(K(\vartheta)|K, v)$ of henselian fields has non-trivial defect if and only if it is immediate. If this holds, we will speak of an Artin-Schreier defect extension. A classification of Artin-Schreier defect extensions (introduced in [11]) distinguishes two types of Artin-Schreier defect extensions, according to their connection with purely inseparable extensions. One type of these extensions can be derived from purely inseparable extensions of degree p by a certain deformation of purely inseparable polynomials into Artin-Schreier polynomials, while the other cannot. The next proposition indicates when such a construction of Artin-Schreier defect extensions is possible; for the proof, see Proposition 4.4 of [11].

Proposition 2.9. *Assume that (L, v) admits an immediate purely inseparable extension of degree p which does not lie in the completion of the field. Then (L, v) admits an Artin-Schreier defect extension.*

Note that if the assumptions on the value group and residue field of (K, v) of (1) hold, then every purely inseparable extension of (K, v) is immediate. Thus if $K^{1/p}$ is not contained in the completion of K , the above proposition yields that (K, v) admits an Artin-Schreier defect extension. However, in the case of p -divisible value group and perfect residue field, we can say much more:

Theorem 2.10. *Assume that (K, v) is a valued field of positive characteristic p with p -divisible value group and perfect residue field. If there is a purely inseparable extension of (K, v) which does not lie in the completion of the field, then K admits an infinite tower of Artin-Schreier defect extensions.*

For the proof see [3], Theorem 1.4.

2.3. Immediate extensions and maximal fields. The henselization and the completion of a valued field are immediate extensions. This together with Theorem 31.21 of [21] gives the following important properties of maximal fields.

Theorem 2.11. *Every maximal field is henselian, complete and defectless.*

Corollary 2.12. *Take a valued field (K, v) such that vK p -divisible if the characteristic of Kv is $p > 0$, and Kv is perfect. Assume that $(M|K, v)$ is an extension such that (M, v) is maximal with vM/vK a torsion group and $Mv|Kv$ algebraic. Then (M, v) is a tame field and hence perfect.*

Proof. Since vK is p -divisible and vM/vK is a torsion group, also vM is p -divisible. Since Kv is perfect and $Mv|Kv$ is algebraic, also Mv is perfect. As every maximal field is defectless by Theorem 2.11, (M, v) is a tame field. \square

For the proof of the next result, see Theorem 31.22 of [21].

Theorem 2.13. *Every finite extension of a maximal field is again a maximal field, with respect to the unique extension of the valuation.*

We are going to show that a field which satisfies the assumptions of (1) cannot be maximal-by-finite. To this end, we need the following result, which is Lemma 2.5 of [11].

Lemma 2.14. *Take an immediate extension $(E|K, v)$, an extension of v from E to \tilde{E} , and a finite defectless subextension $(L|K, v)$ of $(\tilde{E}|K, v)$. If v admits a unique extension from K to L , then $L|K$ is linearly disjoint from $E|K$ and the extension $(L.E|L, v)$ is immediate.*

Lemma 2.15. *Assume that (K, v) is a henselian field which is not maximal, and $(M|K, v)$ is a finite extension such that (M, v) is maximal. Then $(M|K, v)$ is a defect extension, and neither (M, v) nor (K, v) is a tame field.*

Proof. If the extension $(M|K, v)$ were defectless, then Lemma 2.14 would imply that for every nontrivial immediate extension $(E|K, v)$ the extension $(E.M|M, v)$ would be also nontrivial and immediate, contradicting our assumption that (M, v) is a maximal field. As every finite extension of a tame field is defectless, this shows that (K, v) cannot be a tame field. Theorem 2.4 shows that also (M, v) cannot be a tame field. \square

Proof of Theorem 1.5. If (M, v) is maximal and a finite extension of (K, v) , then it follows from Corollary 2.12 that (M, v) is a tame field. Since (K, v) is assumed to be finite, Lemma 2.15 shows that (K, v) must be maximal. Hence it is a defectless field by Theorem 2.11. As it also satisfies **(K1)** and **(K2)** by assumption, it is a tame field, so $(M|K, v)$ is a tame extension. \square

In what follows we will make repeated use of the main theorem of [4]:

Theorem 2.16. *Take a valued field extension $(L|K, v)$ of finite transcendence degree ≥ 0 , with v nontrivial on L . Assume that one of the following four cases holds:*

valuation-transcendental case: vL/vK is not a torsion group, or $Lv|Kv$ is transcendental;

value-algebraic case: vL/vK contains elements of arbitrarily high order, or there is a subgroup $\Gamma \subseteq vL$ containing vK such that Γ/vK is an infinite torsion group and the order of each of its elements is prime to the characteristic exponent of Kv ;

residue-algebraic case: Lv contains elements of arbitrarily high degree over Kv ;

separable-algebraic case: $L|K$ contains a separable-algebraic subextension $L_0|K$ such that within some henselization of L , the corresponding extension $L_0^h|K^h$ is infinite.

Then each maximal immediate extension of (L, v) has infinite transcendence degree over L . If the cofinality of vL is countable (which for instance is the case if vL contains an element γ such that $\gamma > vK$), then already the completion of (L, v) has infinite transcendence degree over L .

The next lemma constitutes the main application of this theorem in the present paper.

Lemma 2.17. *Assume that (K, v) is a henselian field which admits an extension (M, v) of finite transcendence degree such that (M, v) is a maximal field and v is nontrivial on M . Then the maximal separable-algebraic subextension of $M|K$ is a finite extension of K .*

Proof. Applying Theorem 2.16 with $L = M$, its separable-algebraic case shows that $(M|K, v)$ cannot admit an infinite separable-algebraic subextension, since M is its own maximal immediate extension. \square

Corollary 2.18. *Take a separable-algebraic extension $(M|K, v)$ of henselian fields such that (M, v) is a maximal field and v is nontrivial on M . Then the extension is finite and either (K, v) is maximal or $(M|K, v)$ is a nontrivial defect extension.*

Proof. The fact that $M|K$ is a finite extension follows directly from the previous lemma.

Suppose that K is not a maximal field. Then it is maximal-by-finite by the assumption on (M, v) . Therefore, Lemma 2.15 yields that the extension $(M|K, v)$ has nontrivial defect. \square

A valued field (K, v) is called **algebraically maximal** if it does not admit any nontrivial immediate algebraic extension, and **separable-algebraically maximal** if it does not admit any nontrivial immediate separable-algebraic extension. Since henselizations are immediate separable-algebraic extensions, every separable-algebraically maximal field is henselian. In the proof of the following lemma we will make use of the theory of pseudo Cauchy sequences as presented in [8].

Lemma 2.19. *Assume that (K, v) is algebraically maximal. If $(K(x), v)$ is an immediate transcendental extension of (K, v) , then $(K(x), v)$ can be embedded over K in every maximal immediate extension of (K, v) .*

Proof. Since $(K(x), v)$ is an immediate extension of (K, v) , Theorem 1 of [8] yields that x is a pseudo limit of a pseudo Cauchy sequence $(a_\nu)_{\nu < \lambda}$ in (K, v) without a pseudo limit in (K, v) . Since (K, v) admits no nontrivial immediate algebraic extensions, it follows from Theorem 3 of [8] that $(a_\nu)_{\nu < \lambda}$ is of transcendental type. Take any maximal immediate extension (M, w) of (K, v) . Then $(a_\nu)_{\nu < \lambda}$ admits a pseudo limit y in (M, w) by Theorem 4 of [8]. From Theorem 2 of [8] we know that sending x to y induces a valuation preserving isomorphism over K from $(K(x), v)$ to $(K(y), w)$, and thus an embedding of $(K(x), v)$ in (M, w) . \square

2.4. Some facts about completions. By (K^c, v) we will denote the completion of a valued field (K, v) . It is unique up to valuation preserving isomorphism. The following is Lemma 6.25 of [15].

Lemma 2.20. *Take a finite extension $(L|K, v)$ of valued fields. Then there is a unique extension of v from K^c to $L \cdot K^c$ which coincides with v on L . With this extension, $L^c = K^c \cdot L$.*

For the proof of the next lemma we will need the following result which is Theorem 2 of [12].

Theorem 2.21. *Take a valued field (K, v) and the henselization K^h of K with respect to some extension of the valuation v to the algebraic closure of K . Take an element b algebraic over K . If there is an element $a \in K^h$ such that*

$$v(a - b) > v(a - c) \text{ for every } c \in K,$$

then $K(b)$ is not linearly disjoint from K^h over K . In particular, $K(b)$ is not purely inseparable.

Lemma 2.22. *Take a valued field (K, v) and an element η purely inseparable over K . If η does not lie in the completion of K then it also does not lie in the completion of the henselization of K .*

Proof. Assume that η does not lie in the completion of (K, v) . Then there is an element $\gamma \in vK$ such that $v(\eta - c) < \gamma$ for every $c \in K$.

Suppose that η lies in the completion of the henselization K^h of K . Then there is $a \in K^h$ such that $v(\eta - a) > \gamma$. It follows that for every $c \in K$

$$v(\eta - a) > \gamma > v(\eta - c) = v(a - c).$$

But the above theorem shows that this is not possible, as η is purely inseparable over K . \square

3. PROOFS OF THE MAIN RESULTS

Throughout this section, we assume that (K, v) is a valued field satisfying the assumptions (1).

Proposition 3.1. *Assume that (K, v) admits a maximal immediate extension (M, v) of finite transcendence degree. Then K is relatively separable-algebraically closed in M and the relative algebraic closure of K in M is equal to the perfect hull of K .*

Proof. If v is trivial on M , then $M = K$, hence the assertion holds.

Suppose now that v is not trivial on M . As we have already shown in Corollary 2.12, (M, v) is a tame field. Denote by L the relative algebraic closure of K in M . Then Lemma 2.5 shows that also (L, v) is a tame field.

We have that $K^{1/p^\infty} \subseteq M^{1/p^\infty}$. As $vM = vK$ is p -divisible if $\text{char } K = p > 0$, and $Mv = Kv$ is perfect, $(M^{1/p^\infty}|M, v)$ is an immediate extension. Since M is maximal, it follows that $M^{1/p^\infty} = M$ and thus $K^{1/p^\infty} \subseteq L$. By Lemma 2.17, the separable-algebraic extension $L|K^{1/p^\infty}$ must be finite. Since (L, v) is tame, by Theorem 2.4 we obtain that $(L|K^{1/p^\infty}, v)$ is defectless. Since it is an immediate extension of henselian fields, it follows that $L = K^{1/p^\infty}$. This proves that K is relatively separable-algebraically closed in M and that $L = K^{1/p^\infty}$. \square

Proof of Theorem 1.7. Assume that (M, v) is a maximal immediate extension of (K, v) of finite transcendence degree. Take (N, w) to be another maximal immediate extension of (K, v) . It suffices to show that (M, v) can be embedded in (N, w) over K , as a valued field. Indeed, if φ is the embedding, then $(\varphi(M), w)$ is a maximal immediate extension of (K, v) . Since the extension $(N|K, w)$ is immediate, also the subextension $(N|\varphi(M), w)$ is immediate and we obtain that $\varphi(M) = N$. Thus, the fields (M, v) and (N, w) are isomorphic over K .

Consider the family of all subextensions $E|K$ of $M|K$ admitting a valuation preserving embedding in N over K . By Zorn's Lemma, there is a maximal such extension $F|K$; denote its embedding by σ . We wish to show that $F = M$.

Since M, N as maximal fields are henselian, M contains the henselization F^h of F and the embedding extends uniquely to an embedding $\tau: F^h \rightarrow N$. By the maximality of σ we have that $F^h = F$, that is, F is henselian.

Take L to be the relative algebraic closure of F in M . Proposition 3.1 yields that $L = F^{1/p^\infty}$. Assume that $\text{char } K = p$. By Corollary 2.12, (N, w) is perfect. Hence, σ can be extended in a unique way to an embedding τ of L in N . From the maximality of σ it follows that $L = F$, that is, F is relatively algebraically closed in M . Again by Corollary 2.12, (M, v) is tame. Therefore, Lemma 2.5 implies that also (F, v) is a tame field.

Since F is relatively algebraically closed in M , either $F = M$ or the extension $M|F$ is transcendental. Suppose the latter holds. We identify (F, v) with its isomorphic image $(\sigma(F), w)$ in (N, w) . Take an element $x \in M \setminus F$. Then $(F(x)|F, v)$

is an immediate transcendental extension. By Lemma 2.19, the field $F(x)$ can be embedded in (N, w) over F , a contradiction to the maximality of F . Thus we obtain the required equality $M = F$. This proves the first assertion of our theorem.

The second is an immediate consequence of the first. \square

Corollary 3.2. *If (K, v) admits a maximal immediate extension of finite transcendence degree, then it is separable-algebraically maximal.*

Proof. Take an immediate separable-algebraic extension $(E|K, v)$. Then (E, v) is contained in some maximal immediate extension (N, w) of (K, v) . Theorem 1.7 implies that $N|K$ is of finite transcendence degree, hence by Proposition 3.1 the field K is relatively separable-algebraically closed in N . Thus the extension $E|K$ is trivial and consequently, (K, v) admits no proper immediate separable-algebraic extensions. \square

For the proof of Theorem 1.2, we will need the following lemma.

Lemma 3.3. *Assume that (K, v) admits a finite Galois defect extension (E, v) . Then every maximal immediate extension of (E, v) is of infinite transcendence degree over E .*

Proof. Set $L = E \cap K^r$. Lemma 2.2 yields that $(L|K, v)$ is a finite tame extension. Furthermore, vE/vL is a p -group and the residue field extension $Ev|Lv$ is purely inseparable. Since vL is p -divisible and Lv is perfect, as this holds already for the value group and the residue field of (K, v) , the group vE/vL and the extension $Ev|Lv$ are trivial. Thus $(E|L, v)$ is an immediate extension. From this and Proposition 2.8 it follows that

$$[E : L] = d(E|L, v) = d(E|K, v) > 1.$$

This shows that the immediate separable-algebraic extension $(E|L, v)$ is nontrivial. Applying Corollary 3.2 to the field (L, v) in place of (K, v) , we obtain that every maximal immediate extension of (L, v) is of infinite transcendence degree.

Since $(E|L, v)$ is immediate, each maximal immediate extension of (E, v) is also a maximal immediate extension of (L, v) ; since $E|L$ is algebraic, it must be of infinite transcendence degree too. \square

Proof of Theorem 1.2. Note first that by Lemma 2.17, $L|K$ is a finite extension. Take a finite Galois extension E of K containing L . Then $(M.E, v)$, where v is the unique extension of the valuation of M to $M.E$, is again maximal. Furthermore, from the valuation-transcendental case of Theorem 2.16 it follows that vM/vK is a torsion group and the residue field extension $Mv|Kv$ is algebraic. Hence, vM is p -divisible and Mv is perfect, as it holds already for the value group and the residue field of (K, v) . Applying Corollary 2.12 to the extension $M.E|M$, we obtain that $M.E$ is a tame field.

Moreover, since L is relatively-separable algebraically closed in M , the separable-algebraic closure L^{sep} of L is linearly disjoint from M . Since $E|L$ is a subextension of $L^{\text{sep}}|L$, it follows that $L^{\text{sep}} = E^{\text{sep}}$ is linearly disjoint from $M.E$ over E . Hence, E is relatively separable-algebraically closed in $M.E$. Moreover, vE is p -divisible and Ev is perfect. Therefore, it follows from Lemma 2.6 that $(M.E, v)$ is an immediate extension of (E, v) . As $(M.E, v)$ is a maximal field, it is a maximal immediate extension of (E, v) .

Suppose that $(E|K, v)$ were a defect extension. Then by Lemma 3.3, every maximal immediate extension of (E, v) would be of infinite transcendence degree. On the other hand, (M, E, v) is a maximal immediate extension of (E, v) of finite transcendence degree, a contradiction.

We thus obtain that every finite Galois extension of K containing L is defectless. This implies in particular that every finite separable-algebraic extension of K is defectless. Since vK is p -divisible and Kv is perfect, this yields that every finite separable-algebraic extension of K is tame. In particular, $(L|K, v)$ is a finite tame extension. This proves assertions a) and b).

As we have seen, (K, v) admits no separable-algebraic defect extensions. Moreover, by the assumption on value group and residue field of (K, v) , every purely inseparable extension of (K, v) is immediate. Therefore Theorem 2.10 yields that every purely inseparable extension of (K, v) lies in the completion K^c of the field (this is trivial when $\text{char } K = 0$). We have proved assertion c).

Suppose that vM/vK is infinite. Then it is either not a torsion group, or it is an infinite torsion group with all exponents prime to the residue characteristic exponent since vK is p -divisible if $\text{char } Kv = p > 0$. Thus, the valuation-transcendental or the value-algebraic case of Theorem 2.16 applies. Suppose that Mv/Kv is infinite. Then the extension is either transcendental, or it is separable-algebraic since Kv is perfect. Thus, the valuation-transcendental or the residue-algebraic case of Theorem 2.16 applies. In both cases, each maximal immediate extension of (M, v) has infinite transcendence degree over M , which is impossible because (M, v) is itself maximal. This proves assertion d).

Assume from now on that the extension $M|K$ is algebraic. Then $M|L$ is a purely inseparable extension. Moreover, vL is p -divisible and Lv is perfect, as this holds already for the value group and the residue field of (K, v) . Hence the extension $M|L$ is immediate. Together with the fact that (M, v) is a maximal field, this yields that M is a maximal immediate extension of L . From Proposition 3.1 it follows that M is the perfect hull of L .

Before we show the last assertion of part e), we prove the assertion of f). Take a maximal immediate extension (N, v) of (K, v) . As N is henselian, v admits a unique extension to the field $N.L$. Denote this extension again by v . Since the extension $(L|K, v)$ is defectless, Lemma 2.14 implies that $(N.L|L, v)$ is an immediate extension. Furthermore, $N.L$ is a finite extension of N . Thus, by Theorem 2.13 we obtain that $N.L$ is a maximal immediate extension of L . By Theorem 1.7 we obtain that $N.L$ is isomorphic to M over L . Thus in particular, $N.L|L$ is an algebraic extension. Hence the same holds for $N|K$. This means that (K, v) admits a maximal immediate extension algebraic over the field. From Proposition 3.1 it follows that N is equal to the perfect hull of K . Moreover, K^{1/p^∞} is contained in the completion K^c of K , by part c) of our theorem. Since $K^c|K$ is an immediate extension, this yields that $N = K^{1/p^\infty} = K^c$. As the completion (K, v) is unique up to isomorphism, this proves assertion f).

It remains to show that the perfect hull of L is equal to the completion of L . Since $L|K$ is a finite extension, Lemma 2.20 together with assertion f) yield that

$$L^{1/p^\infty} = K^{1/p^\infty}.L = K^c.L = L^c.$$

Therefore, assertion e) holds. □

Proof of Corollary 1.4. Assume that (M, v) is a maximal field. Then by part b) of Theorem 1.2 we obtain that $(M|K, v)$ is a finite tame extension. From Corollary 2.12 it follows that (M, v) is a tame field. Hence, by Theorem 2.4 also (K, v) is a tame field. It remains to show that (K, v) is a maximal field, but this follows directly from Lemma 2.15. \square

Proof of Theorem 1.8. If a) holds, then the assertion of the theorem follows from part a) of Theorem 1.2, as separably-tame fields admit no separable-algebraic defect extensions.

Assume that b) holds. Then the assertion of the theorem is a consequence of part c) of Theorem 1.2. Furthermore, by Theorem 2.10 it admits also separable-algebraic defect extensions, hence the condition of a) is satisfied. \square

Assume that (K, v) satisfies the assumptions of part b) of Theorem 1.8. Then we can give an explicit construction of an immediate extension of (K, v) of infinite transcendence degree. Indeed, if the perfect hull of (K, v) is not contained in the completion of K , then Theorem 2.10 yields that (K, v) admits a separable-algebraic extension $L|K$ which is an infinite tower of Artin-Schreier defect extensions. Now by the separable-algebraic case of Theorem 2.16 we obtain that every maximal immediate extension of (L, v) is of infinite transcendence degree. As every Artin-Schreier defect extension is immediate, we deduce that $(L|K, v)$ is immediate and any maximal immediate extension of L is also a maximal immediate extension of K . This proves the assertion of Theorem 1.8. Furthermore, the proof of Theorem 2.10 presents a possible construction of the tower of Artin-Schreier defect extensions $(L|K, v)$ (see the proof of Theorem 1.4, [3]). Also the proof of Theorem 2.16 shows how to construct the immediate extension of infinite transcendence degree of (L, v) (cf. Theorem 1.1 of [4]). This gives us a construction of an immediate extension of (K, v) of infinite transcendence degree.

Remark 3.4. Note that if $(L|K, v)$ is a finite separable extension, then the perfect hull of K is contained in the completion K^c of K if and only if the same holds for L . Indeed, if $K^{1/p^\infty} \subseteq K^c$, then by Lemma 2.20 we have that

$$L^{1/p^\infty} = K^{1/p^\infty} \cdot L \subseteq K^c \cdot L = L^c.$$

Conversely, assume that $L^{1/p^\infty} \subseteq L^c$. Then in particular, $K^{1/p^\infty} \subseteq L^c$. By Lemma 2.20 $L^c = K^c \cdot L$. Hence, $L^c|K^c$ is a separable algebraic extension, as $L|K$ is. We thus deduce that $K^{1/p^\infty} \subseteq K^c$.

This shows that we can replace condition b) of Theorem 1.8 by the equivalent condition:

2') for some finite separable extension L of K , the perfect hull of L is not contained in the completion of L .

Remark 3.5. Note that in Theorem 1.7 we can omit the assumption that (K, v) is a henselian field. This follows from the fact that the henselization of a valued field is unique up to isomorphism and that every maximal immediate extension contains a henselization.

We show that also in Theorem 1.8 we can omit the assumption that (K, v) is a henselian field. Condition a) should then read as follows:

a') There is a finite separable-algebraic extension $(F|K, v)$ such that the valuation v extends in a unique way from K to F and $(F|K, v)$ has nontrivial defect.

Take a maximal immediate extension (M, v) of (K, v) . Then M contains a henselization K^h of K . Assume that condition a') is satisfied and denote by F^h the henselization of F with respect to the unique extension of the valuation of M to $M.F$. Then equation (6) shows that also the extension $F^h|K^h$ has nontrivial defect. Thus from Theorem 1.8 we deduce that every maximal immediate extension of K^h is of infinite transcendence degree. Hence also $M|K$ has infinite transcendence degree.

Now take a valued field (K, v) such that the perfect hull of K is not contained in the completion of K . Take a maximal immediate extension (M, v) of (K, v) . Then M contains a henselization K^h of K , the completion K^c of K and the completion $(K^h)^c$ of K^h . Since by assumption K admits a purely inseparable extension which is not contained in K^c , Lemma 2.22 yields that the extension is also not contained in $(K^h)^c$. Thus K^h is a henselian field which satisfies the assumptions of Theorem 1.8. We therefore obtain that every maximal immediate extension of K^h is of infinite transcendence degree. Since (M, v) is a maximal immediate extension of K^h , we deduce that also $M|K$ has infinite transcendence degree. Moreover, from Theorem 2.10 it follows that (K, v) satisfies condition a').

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